First a note about multiplication:

(Multiplication on right of matrix)

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + z\begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

"column operation"

EX
$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (-2)\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

answer = 3·column 1 − 2·column 2 $= \begin{bmatrix} 3-8 \\ 6-10 \\ 9-12 \end{bmatrix}$

(Multiplication on left of matrix)

$$\begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} = x\begin{bmatrix} 1 & 4 & 7 \end{bmatrix} + y\begin{bmatrix} 2 & 5 & 8 \end{bmatrix} + z\begin{bmatrix} 3 & 6 & 9 \end{bmatrix}$$

"row operation"

EX:
$$\begin{bmatrix} 3 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} = 3\cdot\begin{bmatrix} 1 & 4 & 7 \end{bmatrix} - 2\cdot\begin{bmatrix} 2 & 5 & 8 \end{bmatrix}$$

answer = 3·row 1 − 2·row 2

$$= \begin{bmatrix} 3-4 & 12-10 & 21-16 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 2 & 5 \end{bmatrix}$$

Row operations are a basic part of the LU decomposition.

LU-Decomposition $A = L \cdot U$

- U is upper Δ, same shape as A
- L is lower Δ, 1 on diagonal, square.

Compute the LU-decomposition by writing a series of products $= A$ which slowly become more triangular in shape.

EX
$$\begin{bmatrix} 2 & -1 & 3 \\ 4 & -3 & 8 \\ -2 & 0 & -3 \end{bmatrix}$$

(begin with "identity matrix" here)

Note: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = A$ (Matrix-version of number 1)

(begin with A here)

[Step 1: Write matrices which will become L and U]
$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{will be } L}\underbrace{\begin{bmatrix} 2 & -1 & 3 \\ 4 & -3 & 8 \\ -2 & 0 & -3 \end{bmatrix}}_{\text{will be } U}$$

Plan: Convert right-side matrix to be more triangular. Begin with column 1. $\Longrightarrow$ Move 4 and −2 to matrix L

$$\begin{bmatrix} 1\cdot 2 & -1 & 3 \\ 4/2\cdot 2 & & \\ -2/2\cdot 2 & & \end{bmatrix} \underset{\text{check}}{=} \begin{bmatrix} 1 & 0 & 0 \\ 4/2 & 1 & 0 \\ -2/2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 \\ 0 & & \\ 0 & & \end{bmatrix}$$

"pivot"

Note: When 4 & −2 move to L, you must divide by 2 (the "pivot") so that product is correct.

Step 2: Move "non-triangular" elements from column 1 of U to L

→ divide them by the "diagonal" element in col.

"diagonal" element in column is "pivot"

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 0 & & \\ 0 & & \end{bmatrix}$$

Note: Row 1 of U is not changed.

When an element moves from U to L, this also changes the other elements in its row.

→ Row 2 of $\begin{bmatrix} 1 & 0 & 0 \\ \boxed{2 \quad 1 \quad 0} \\ -1 & 0 & 1 \end{bmatrix} U = 2(\text{row 1 of } U) + 1\cdot(\text{row 2 of } U)$

So $\begin{pmatrix}\text{new} \\ \text{row 2}\end{pmatrix} = -2\cdot\text{row 1} + \text{row 2}$

reverse of [2 1 0]

and $\begin{pmatrix}\text{new} \\ \text{row 3}\end{pmatrix} = -(-1)\cdot\text{row 1} + \text{row 3}$

reverse of [-1 0 1]

Multiplying by L is doing row operations — so U must be modified with reverse operations

Step 3: "Correct" the rows of elements that moved out of U:

$$\begin{pmatrix}\text{new} \\ \text{row}\end{pmatrix} = -\begin{pmatrix}\text{Multiplier} \\ \text{from } L\end{pmatrix}\cdot\begin{pmatrix}\text{Pivot's} \\ \text{Row}\end{pmatrix} + \begin{pmatrix}\text{old} \\ \text{row}\end{pmatrix}$$

Value in L of element that was removed from U

Row of U with pivot

(Row of U corresponding to Column of multiplier)

"Pivot"

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 0 & -2(-1)+3 & -2\cdot 3+8 \\ 0 & -(-1)(-1)+0 & -(-1)\cdot 3-3 \end{bmatrix}$$ ← "Pivot Row"

"Multipliers"

Row 2 = $-2\,[2 \;\; -1 \;\; 3]$ ← "Pivot Row"
$+\,[4 \;\; -3 \;\; 8]$ ← Old Row 2

and Row 3 = $-(-1)\,[2 \;\; -1 \;\; 3]$ ← "Pivot Row"
$+\,[-2 \;\; 0 \;\; -3]$ ← Old Row 3

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & 2 \\ 0 & -1 & 0 \end{bmatrix}$$

Step 4: Move to next column & repeat 2–3

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & 2 \\ 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & -1\cdot 2+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix} \overset{??}{=} \begin{bmatrix} 2 & -1 & 3 \\ 4 & -3 & 8 \\ -2 & 0 & \boxed{-3} \end{bmatrix}$$

To check your answer, you can compute a few terms in product.

Example computations

$$\begin{bmatrix}3&-2&1\\-6&6&-3\\9&-10&3\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}3&-2&1\\-6&6&-3\\9&-10&3\end{bmatrix}$$

$$= \begin{bmatrix}1&0&0\\-2&1&0\\3&0&1\end{bmatrix}\begin{bmatrix}3&-2&1\\0&-(-2)(-2)+6&-(-2)\cdot 1-3\\0&-3(-2)-10&-3\cdot 1+3\end{bmatrix}$$

$$= \begin{bmatrix}1&0&0\\-2&1&0\\3&0&1\end{bmatrix}\begin{bmatrix}3&-2&1\\0&2&-1\\0&-4&0\end{bmatrix}$$

$$= \begin{bmatrix}1&0&0\\-2&1&0\\3&-2&1\end{bmatrix}\begin{bmatrix}3&-2&1\\0&2&-1\\0&0&-(-2)(-1)+0\end{bmatrix}$$

$$= \begin{bmatrix}1&0&0\\-2&1&0\\3&-2&1\end{bmatrix}\begin{bmatrix}3&-2&1\\0&2&-1\\0&0&-2\end{bmatrix}$$

Check:

$$\begin{bmatrix}3&-2&1\\-6&\boxed{6}&-3\\9&-10&\boxed{3}\end{bmatrix} \overset{?}{=} \begin{bmatrix}1&0&0\\-2&1&0\\3&-2&1\end{bmatrix}\begin{bmatrix}3&-2&1\\0&2&-1\\0&0&-2\end{bmatrix}$$

→ Just compute a few elements to check

$$3 \overset{?}{=} 3\cdot 1 + (-2)(-1) + 1(-2) \quad \text{ok.}$$

$$6 \overset{?}{=} -2(-2) + 1\cdot 2 + 0\cdot 0 \quad \text{ok.}$$

$$\begin{bmatrix}3&3&1\\-6&-4&-1\\9&5&1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}3&3&1\\-6&-4&-1\\9&5&1\end{bmatrix}$$

$$= \begin{bmatrix}1&0&0\\-2&1&0\\3&0&1\end{bmatrix}\begin{bmatrix}3&3&1\\0&2&1\\0&-4&-2\end{bmatrix}$$

$$= \begin{bmatrix}1&0&0\\-2&1&0\\3&-2&1\end{bmatrix}\begin{bmatrix}3&3&1\\0&2&1\\0&0&0\end{bmatrix}$$

$$\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix}$$

$$= \begin{bmatrix}1&0&0\\-1/2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}2&-1&0\\0&3/2&-1\\0&-1&2\end{bmatrix}$$

$$= \begin{bmatrix}1&0&0\\-1/2&1&0\\0&-2/3&1\end{bmatrix}\begin{bmatrix}2&-1&0\\0&3/2&-1\\0&0&4/3\end{bmatrix}$$

Note: "4x4 negative 2nd derivative matrix"

$$\begin{bmatrix}2&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&2\end{bmatrix} = \begin{bmatrix}1&0&0&0\\-1/2&1&0&0\\0&-2/3&1&0\\0&0&-3/4&1\end{bmatrix}\begin{bmatrix}2&-1&0&0\\0&3/2&-1&0\\0&0&4/3&-1\\0&0&0&5/4\end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & -2 & 2 \\ 9 & 1 & -5 & 6 \\ 3 & 5 & -4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & -2 & 2 \\ 9 & 1 & -5 & 6 \\ 3 & 5 & -4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & -2 & 2 \\ 0 & -2 & 1 & 0 \\ 0 & 4 & -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & -2 & 2 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 & -1 & 1 \\ 4 & -4 & 1 & 3 \\ -4 & 4 & 11 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & -1 & 1 \\ 4 & -4 & 1 & 3 \\ -4 & 4 & 11 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & -1 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 9 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & -1 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Note: When an element moves to L, it stays in the same row... but it could move to a different column!
Elements move to the column which is multiplied by the pivot element. When the pivot is on the diagonal, then this is the same column.

$$\begin{bmatrix} 2 & -3 & -3 \\ -4 & 6 & 9 \\ 6 & -9 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 & -3 \\ -4 & 6 & 9 \\ 6 & -9 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 & -3 \\ 0 & 0 & 3 \\ 0 & 0 & 6 \end{bmatrix} \leftarrow \text{pivot}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 & -3 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

(Since the pivot was in row 2, the $6/3$ moves to column 2.)

In the previous two examples, we did more than just make U be upper triangular. We put U in "stair-step" or "reduced" form.

Recall: "Pivots" are the first nonzero elements in their row.

Def: A matrix is "reduced" if lower rows have pivots further to the right.

Ex:

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Not reduced (pivots are circled)

Row 2 pivot is same col as Row 1

$$\begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Not reduced

Row 2 pivot is left of Row 1

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Not reduced

Row 3 pivot is same col as Row 2

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

Reduced

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

Reduced

Upper triangular matrices with nonzero diagonals are all reduced

⤳ This is the usual form of U.

(But not always.)

LU-decompositions are unpleasant when A has more rows than columns. Remember, L is square with the same number of rows as A... If A has many rows, then L could be much larger than A!

$$\begin{bmatrix} 2 & 3 \\ -4 & -7 \\ 6 & 2 \\ 8 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -4 & -7 \\ 6 & 2 \\ 8 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & -1 \\ 0 & -7 \\ 0 & -11 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & 7 & 1 & 0 \\ 4 & 11 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 2 & 3 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The matrix L is big and annoying to write (although only the first two columns contain any information...)